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On routing of wavebands for all-to-all communications in all-optical paths and cycles[☆,☆☆]

Michele Flammini[a,*], Alfredo Navarra[a], Andrzej Proskurowski[b]

[a] *Dipartimento di Informatica, University of L'Aquila, Via Vetoio loc. Coppito, I-67100 L'Aquila, Italy*
[b] *Computer and Information Science Department, University of Oregon, Eugene, OR 97403, USA*

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Abstract

We discuss a model of the all-optical communication technology and an implementation of a simple task, all-to-all, in simple topologies like paths and cycles. The model assumes a single interval (variant of band-pass) filter extracting signal wavelengths for processing and forwarding in intermediate communication nodes. In an attempt to use a minimum number of wavelengths, we give lower and upper bounds on the cardinality of the spectrum used in four cases arising from different assumptions on the topology and the filters. In particular, we propose efficient schedules of directed paths between all pairs of nodes in graphs of maximum node degree two, under the assumption of either a "linear" or "wrapped-around" wavelength spectrum.

Keywords: All-optical networks; WDM; Wavebands filtering; Routing; Paths; Cycles

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* Corresponding author. Tel.: 39 0862 433730; fax: 39 0862 433057.
E-mail addresses: flammini@di.univaq.it (M. Flammini), navarra@di.univaq.it (A. Navarra), andrzej@cs.uoregon.edu (A. Proskurowski).

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1. Introduction

All-optical networks have been investigated in recent years largely due to the promise of data transmission rates several orders of magnitude higher than current networks [4,5,10,11,14–16]. Major applications are in video conferencing, scientific visualization and real-time medical imaging, high-speed supercomputing and distributed computing [10,16,6].

The key to high speeds in all-optical networks is to maintain the signal in optical form, thereby avoiding the prohibitive overhead of conversion to and from the electrical form at the intermediate nodes. The width of the optical spectrum (bandwidth, the number of wavelengths used) of the optical fiber is utilized through *wavelength-division multiplexing*: two signals connecting different source–destination pairs may share a link, provided they are transmitted on carriers having different wavelengths (or colors) of light. The bandwidth is a scarce resource, so that given a communication task and a nxetwork topology, the communication protocol is often designed to minimize the total number of used colors. This number is trivially bounded from below by the maximum load, which is the maximum number of connection paths sharing the same physical edge, but it can be strictly higher. Unfortunately, its minimization is computationally hard, even for specific networks like trees and rings [7]. A survey of the main results related to this problem can be found in [1,9].

In this paper, we are concerned with the simple topologies of paths and cycles. In paths, due to the equivalence with the coloring problem in interval graphs, for every communication task it is possible to provide in polynomial time a solution using the number of colors equal to the maximum load [2], while in cycles and trees there are solutions with the number of colors at most r times the maximum load with $r = \frac{3}{2}$ [13] and $r = \frac{5}{3}$ [12], respectively. In the all-to-all communication task, in which connecting paths must be established for all the possible source–destination pairs, it is always possible to attain the maximum load in trees [8], thus yielding $\lceil (n^2 - 1)/4 \rceil$ colors for paths, and in cycles, giving the $\lceil (n^2 - 1)/8 \rceil$ upper bound independently proved in [17,3]. The same holds for many other networks and no topology is known that requires a higher number of colors (see [3,1,9]).

In this paper we consider routing elements that are capable of directing intervals of contiguous wavelengths ("wavebands"). Grouping signals in intervals, rather than selecting individual wavelengths, simplifies the hardware requirements of the routing nodes. In fact, the number of optical filters and switches needed is then equal to the number of wavebands instead of single wavelengths (see for instance [16]).

The main point of this paper is that an increased complexity of the filtering hardware is not likely to significantly decrease the total number of colors necessary to complete the all-to-all communication task.

The paper is organized as follows. Section 2 introduces the necessary notation and definitions. Sections 3 and 4 present our results concerning the number of colors needed to perform all-to-all communications respectively in paths and cycles under different assumptions on the band-pass filters. Finally, Section 5 presents concluding remarks and discusses some open questions.

2. Definitions and models

We model the topology of a network by an undirected graph $G = (V, E)$ with $n = |V|$ nodes and $m = |E|$ edges. Each $x \in V$ represents a router and each undirected edge $\{x, y\} \in E$ between two nodes $x, y \in V$ represents the two directional links between the corresponding routers (from x to y and from y to x).

In all-optical communication networks, sites (nodes) communicate by sending signals of different wavelengths (colors, frequencies) along optical fiber lines (directed edges). A point-to-point communication requires establishing a uniquely colored directed path between the two nodes (a sequence of directed edges such that adjacent edges share end-nodes), where the color is different from the colors labeling all the other paths sharing an edge in the same direction. All the colors used in a communication task are ordered (forming the "frequency spectrum"); we will call such an order *circular* when the first and the last colors are considered adjacent, and *linear* otherwise. Our motivation for introducing the circular spectrum model is given below.

We assume that an end-node of an edge receives (consumes) some signals transmitted in its direction through the edge, and forwards (hands over) all the other signals. This discrimination is achieved by means of filtering: in our model, the colors received by a node at an incoming edge (respectively, forwarded to an outgoing edge) belong to an interval disjoint from all of the other intervals selected in the node (the filters are of the band-pass type). This assumption has a limiting effect on the parts of the spectrum consumed (respectively, handed over) by intermediate nodes. For instance, at a node of degree 2, one interval of colors arriving from an incoming edge is received and another, a disjoint one, is forwarded along the next edge.

In the linear case, the received and forwarded intervals are, respectively, in the lower and higher ends of the spectrum, or vice versa (the filters are of low- and high-pass type.) In the circular case, if the received colors form an interval, then the remaining parts of the spectrum forms a circular interval and as such can be handed over and forwarded under our model, and a symmetric situation holds when an interval of the spectrum is forwarded so that all the remaining parts of the spectrum can be received (using band-pass filters). This observation extends as well to higher degree nodes, where a single interval of colors is routed along each edge and a single interval is received.

Notice that the circularity of the spectrum can be simulated with only an additional pass-band filter per incoming edge, as the (unique) circular interval of the edge can be split in two linear ones, thus increasing the overall number of linear intervals of one. Another possibility might be that of substituting one standard pass-band filter with a negative one that blocks instead of selecting a given waveband, thus forwarding a circular interval of colors. Besides its technological feasibility, as we will show in the sequel, the circular spectrum assumption is fundamental under a characterization point of view, as it corresponds to the minimum increase of hardware complexity allowing universality, that is the possibility of implementing any given communication pattern.

We are concerned here with the all-to-all communication task where a connecting colored dipath must be established for each possible source–destination pair.

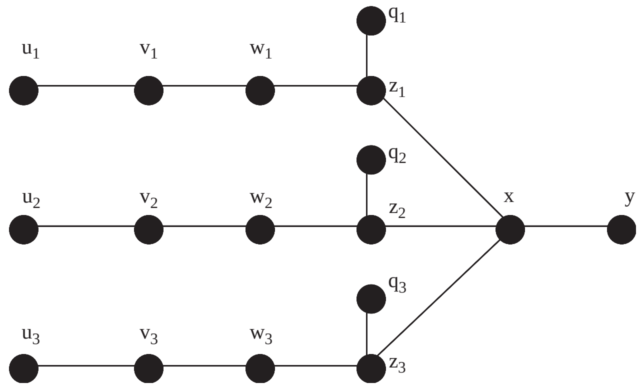

Fig. 1. A network not allowing all-to-all communications under the linear spectrum assumption.

Definition 2.1. Given a network G, $l(G)$ (resp. $c(G)$) is the minimum number of colors needed to perform all-to-all communications in G routing single intervals under the linear (resp. circular) spectrum assumption.

The constraint that each node can receive and forward single intervals, in general, causes an increase of the number of necessary colors. Worse than that, there are cases in which it is even impossible to realize given communication tasks.

Lemma 2.2. *There exists a network G in which it is not possible to perform all-to-all communications by routing single intervals under the linear spectrum assumption.*

Proof. Consider the tree network in Fig. 1.

Let a_1, b_1, c_1 (resp. a_2, b_2, c_2 and a_3, b_3, c_3) denote the colors used by node y to reach nodes u_1, v_1, w_1 (resp. u_2, v_2, w_2 and u_3, v_3, w_3). Since the edge from y to x carries single intervals of the linear spectrum, there must be 3 disjoint linear intervals such that the first one contains the colors a_1, b_1, c_1, the second one the colors a_2, b_2, c_2 and the third one the colors a_3, b_3, c_3. Without loss of generality, we can assume that the first interval in the spectrum comes before the second one and the second one comes before the third, i.e., the middle interval contains a_2, b_2 and c_2. Moreover, c_1 cannot be located between a_1 and b_1, since otherwise by the linearity of the spectrum either a_1 or b_1 cannot be forwarded after the receiving interval of w_1. Therefore, either b_1 is between a_1 and c_1 or a_1 is between b_1 and c_1. Without loss of generality let us assume that $a_1 < b_1 < c_1$, and similarly that $a_2 < b_2 < c_2$ and $a_3 < b_3 < c_3$. Recalling also the order between the intervals, we then have $a_1 < b_1 < c_1 < a_2 < b_2 < c_2 < a_3 < b_3 < c_3$.

Let d_1, d_2 and d_3 be the colors used by node q_2 to reach v_1, v_2 and v_3 respectively. Then we must have $a_1 < d_1 < c_1$ and similarly $a_2 < d_2 < c_2$ and $a_3 < d_3 < c_3$, so that $d_1 < d_2 < d_3$. Since q_2 reaches v_2 through the edge from z_2 to w_2 and v_1 and v_3 through the edge from z_2 to x, at z_2 one linear interval containing d_1 and d_3 but not d_2 should be forwarded from $\{q_2, z_2\}$ to $\{z_2, x\}$, which is clearly impossible.

Therefore, it is not possible to perform all-to-all communications in G under the linear spectrum assumption. □

However, better results hold for the circular spectrum model.

Lemma 2.3. *For any network G it is possible to perform all-to-all communications by routing single intervals under the circular spectrum assumption with $c(G) \leqslant n(n-1)$ colors.*

Proof. Let T be any spanning tree of G and choose an arbitrary node r as the root of T. Starting at r, assign n distinct labels from 1 to n to the nodes of T in a preorder fashion and let $label(v)$ be the label of node v in T.

Given an edge $\{u, v\}$ of T, let $T(u, v)$ be the subtree of T induced by the nodes reachable from u through $\{u, v\}$. Then, by the preorder labeling, the labels assigned to $T(u, v)$ form a contiguous (possibly circular) interval of labels. This is trivially true if u is the parent of v; otherwise, it follows by observing that in the circular spectrum model the complement of an interval is also an interval and the labels of $T(u, v)$ constitute the complement of the interval of the labels of $T(v, u)$.

To schedule all-to-all communications in T, we assign a disjoint receiving interval I_u of $n-1$ colors to each node u of T, in such a way that for any incident edge $\{u, v\}$, each node of $T(u, v)$ uses a different color in I_u to communicate with u. Intervals are assigned in the order dictated by the preorder labels, so that the receiving interval of the node with label i, $1 \leqslant i \leqslant n$, is $[(i-1)(n-1)+1, i(n-1)]$. Therefore, the total number of used colors is $n(n-1)$, and therefore $c(G) \leqslant n(n-1)$.

It remains to show that such an assignment is feasible for the circular spectrum model. We observe first that for any edge $\{u, v\}$ of T, since the labels of $T(u, v)$ form an interval, also the union of all the receiving intervals of the nodes in $T(u, v)$ form an interval of the circular spectrum.

By the above observation, given any edge $\{u, v\}$ with the other neighbors of v (if any) denoted as $w_1, \ldots, w_k$, the signals traversing $\{u, v\}$ toward v can be routed as follows: the interval of colors I_v is received by v, while for every i, $1 \leqslant i \leqslant k$, the interval resulting from the union of all the receiving intervals of the nodes in the subtree $T(v, w_i)$ is forwarded along $\{v, w_i\}$. This concludes the proof. □

An interesting problem is that of reducing the number of colors established in the construction of the above lemma. A trivial lower bound is given by the maximum load $load(G)$, which is the maximum number of directed paths that share the same edge, as all such paths must use different colors. In the special case of the star graph ($K_{1,n-1}$) it is possible to obtain $c(G) = n-1$, thus attaining the maximum load $load(G) = n-1$. Without the waveband interval constraint, this lower bound can always be attained in the case of all-to-all communications in an arbitrary tree [8]. However, the following lemma states that in the circular case the ratio between $c(G)$ and $load(G)$ can grow proportionally to the number of nodes.

Lemma 2.4. *For arbitrarily large values of n there exists a tree T with n nodes such that $c(T)/load(T) \geqslant (n-2)/16$.*

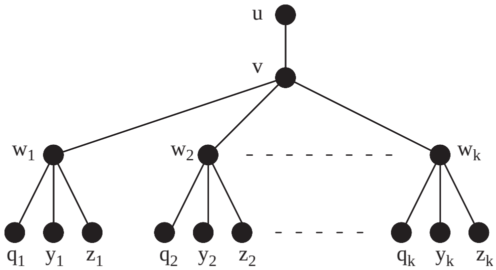

Fig. 2. Tree with $c(T)/load(T) \geqslant (n-2)/16$.

Proof. Given an integer $k > 0$, consider the tree $T=(V,E)$ with $V = \{u\} \cup \{v\} \cup \{w_i, q_i, y_i, z_i | 1 \leqslant i \leqslant k\}$ and $E = \{\{u,v\}\} \cup \{\{v, w_i\} | 1 \leqslant i \leqslant k\} \cup \{\{w_i, q_i\}, \{w_i, y_i\}, \{w_i, z_i\} | 1 \leqslant i \leqslant k\}$ (see Fig. 2).

In the above tree, we have $k = (n-2)/4$ branches (subtrees induced by the 4 nodes w_i, q_i, y_i, z_i for each i, $1 \leqslant i \leqslant k$).

Clearly, the maximum load is $load(T) = 4(n-4)$ and is attained in the edges $\{v, w_i\}$, $1 \leqslant i \leqslant k$. In order to provide a lower bound on $c(T)$, we will determine the number of wavelengths needed for all nodes q_i, y_i, z_i to be reached by all the other nodes outside their branch.

For each i, $1 \leqslant i \leqslant k$, let I_{q_i}, I_{y_i} and I_{z_i} be the minimal (with respect to inclusion) receiving subintervals of q_i, y_i and z_i, respectively, that contain the colors of all the directed paths reaching them from outside their branch. Since all such paths share the edge from v to w_i, I_{q_i}, I_{y_i} and I_{z_i} must be disjoint. Therefore, because at node v the colors of the directed paths from u to the nodes q_i, y_i, z_i have to be included into a forwarding interval I_i toward the edge $\{v, w_i\}$, one of the three subintervals, say I_{y_i}, must be properly contained in I_i. As a consequence, since I_{y_i} contains at least $n-4$ colors and all the I_is have to be mutually exclusive, $c(T) \geqslant k(n-4) = (n-2)(n-4)/4$, from which follows the lemma. □

Notice that the above result is asymptotically optimal. In fact, given any tree T, exploiting Lemma 2.3 and the fact that $load(T) \geqslant n-1$, the inequality $c(T)/load(T) \leqslant n$ always holds.

In the remainder of this paper, we consider the simple cases of graphs with maximum node degree 2. We will provide upper and lower bounds on $l(G)$ and $c(G)$ when G is a path or a cycle of nodes.

3. Gossiping in paths

In a path network $G = P_n$ of length $n-1$, nodes are numbered from 0 to $n-1$ and are arranged consecutively, so that for each i, $0 \leqslant i < n-1$, node i is adjacent to node $i+1$. Since the transmissions in one direction are independent from those in the other, we can focus on only one direction, say in the increasing order of node numbers, and use the symmetric solution in the other direction.

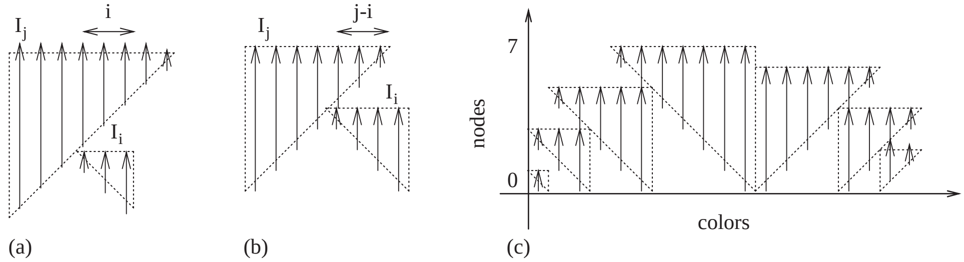

Fig. 3. (a) Overlap when $\min(i, j-i) = i = 3$. (b) Overlap when $\min(i, j-i) = j-i = 3$. (c) The linear spectrum construction for P_8.

Before treating the linear and circular cases in detail, let us observe that a trivial spectrum allocation for all-to-all communications in P_n is to assign disjoint receiving intervals I_i to each node i, $1 \leqslant i \leqslant n-1$, in such a way that $I_i = [i(i-1)/2+1, i(i+1)/2]$. This clearly gives a total number of $n(n-1)/2$ colors. Such an assignment is feasible with respect to our filtering constraints, as it is easy to see that the interval I_i of the colors (subinterval of those transmitted along the edge $\{i-1, i\}$) is received at node i, and the interval $[i(i+1)/2+1, n(n-1)/2]$ is forwarded by node i to the nodes j greater than i. Therefore, $c(G) \leqslant l(G) \leqslant n(n-1)/2$.

The key idea to reduce the number of colors is the reuse of colors, as follows. Given any two nodes i and j, $i < j$, it is possible to overlap I_i and I_j for a maximum saving (the number of reused colors) of $\min(i, j-i)$ colors (see Fig. 3).

According to the spectrum assumption, different overlapping configurations of intervals are possible. Let us thus consider the linear and circular cases separately.

3.1. Linear spectrum model

The obvious lower bound on the number of colors $l(G)$ needed to perform all-to-all communications in a network G under the linear spectrum assumption is the maximum edge load $load(G)$. Since in a path P_n the edge from $i-1$ to i, $1 \leqslant i < n$, has to carry transmissions between every node pair j and k such that $0 \leqslant j < i \leqslant k < n$, the load of this edge is $i(n-i)$, for a maximum of $\lceil n/2 \rceil \lfloor n/2 \rfloor$ in one of the middle edges of the path. It is well known that in an arbitrary topology, the lower bound provided by the maximum load may have to be exceeded (see, for instance, [9]). Because of our filtering constraint, even in paths there are cases in which this lower bound cannot be attained.

Lemma 3.1. *For $n \geqslant 5$, $l(P_n) \geqslant n(n-1)/2 - 2n + 7$.*

Proof. By the linearity of the spectrum, at any given node i, $0 < i < n-1$, the interval of the forwarded colors is located either before the received interval I_i or after it. In the former case we say that node i and its received interval are of *type* 0, in the latter case of *type* 1 . Notice that each I_i with $0 < i < n-1$ can be overlapped only with the receiving interval of the previous node of the same type and of the next node of the same type, since

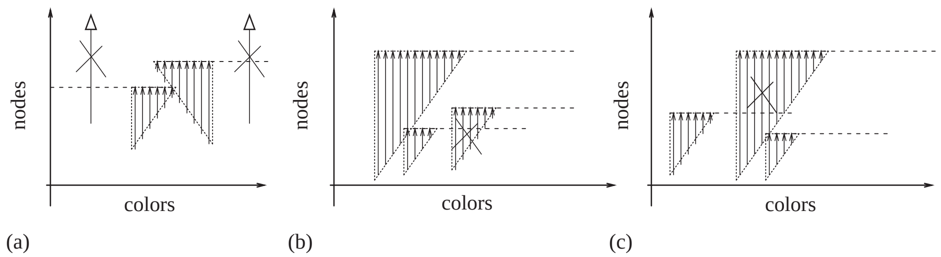

Fig. 4. Impossibility of overlapping among (a) intervals of different types and (b,c) intervals of the same type that are not adjacent.

their forwarding intervals have to overlap to let the preceding nodes reach all the succeeding ones (see Fig. 4).

For what concerns the terminal node $n-1$, since no color is forwarded after it, I_{n-1} can be overlapped both with the closest preceding interval of type 0 and with the closest preceding one of type 1. If I_{n-1} overlaps with just one interval I_i with $i < n-1$, then $n-1$ and I_{n-1} are considered of the same type of i and I_i. If such condition does not hold, that is I_{n-1} overlaps with two intervals, since one of them is I_{n-2} and the other some I_i with $i < n-2$, I_{n-1} can be shifted so as to overlap only with I_i without increasing the total number of colors, thus falling again in the previous case.

We now show that the number of reused colors among the receiving intervals I_i with $2 \leqslant i \leqslant n-1$ is at most $2n-8$. Then, since I_1 has size 1 and its overlapping with the successive intervals allows us to save only another color, all the savings total up to at most $2n-7$. The lemma follows by subtracting this amount from the sum of the sizes of all the receiving intervals, i.e., $n(n-1)/2$.

Let us say that two nodes of the same type are *nearby* if all the nodes between them are of the other type. Then, by the above discussion, listing the types of the nodes from 2 to $n-1$ defines a binary sequence such that the distance between any pair of nearest identical characters in the sequence, representing a pair of nearby nodes i and j with $i < j$ of the same type, is an upper bound on the savings of colors by the overlap of I_i and I_j.

Given two nearby nodes i and j with $i < j$, the length of the subpath connecting i and j is thus equal to the maximum number of savings due to the overlapping of I_i and I_j. Let $\mathcal{P}$ be the set of all such subpaths. Then, the sum of the lengths of all the subpaths in $\mathcal{P}$ is an upper bound on the maximum possible total savings among the receiving intervals I_i with $2 \leqslant i \leqslant n-1$. Since each edge having both the endpoints between 3 and $n-2$ can belong to at most two paths of $\mathcal{P}$, while the edges $\{2, 3\}$ and $\{n-2, n-1\}$ belong to at most one path, the total of the path lengths is at most $2(n-5)+1+1 = 2n-8$, hence the lemma follows. $\square$

In order to get a construction attaining such bound, starting with the trivial construction described at the beginning of this section in which each node i with $1 \leqslant i \leqslant n-1$ receives a disjoint interval of colors $I_i = [i(i-1)/2+1, i(i+1)/2]$, let us show how to overlap the receiving intervals so that the assignment remains compatible with the linear spectrum assumption and $2n-7$ colors are saved.

In the new scheme, the receiving intervals I_i alternate in such a way that the intervals of the odd nodes are of *type* 1 and are located before the receiving intervals of the even nodes, which are of *type* 0. Moreover, intervals are overlapped in such a way that I_3 contains I_1 saving one wavelength and for $i > 3$, I_i saves two wavelengths through overlapping with I_{i-2} (see Fig. 3(c)). More precisely,

$$I_1 = [1, 1], \quad I_2 = \left[\frac{n(n-1)}{2} - 2n + 6, \frac{n(n-1)}{2} - 2n + 7\right], \quad I_3 = [1, 3]$$

and

$$I_i = \left[\frac{(i-1)^2}{4} + 3 - i, \frac{(i-1)^2}{4} + 2\right]$$

if i is odd and

$$I_i = \left[\frac{n(n-1)}{2} - 2n + 6 - \frac{i^2 - 2i}{4}, \frac{n(n-1)}{2} - 2n + 5 - \frac{i^2 - 6i}{4}\right]$$

for values of $i > 3$. Such an assignment is clearly feasible for the linear spectrum model, because odd nodes i forward the linear interval consisting of all the wavelengths after I_i, while even nodes i forward all the wavelengths before I_i.

Therefore, the following theorem has been proved.

Theorem 3.2. *For* $n \geqslant 5$, $l(P_n) = n(n-1)/2 - 2n + 7$.

3.2. Circular spectrum model

In the circular case, more overlapping configurations are possible, since all the colors before and after a receiving interval I_i can be forwarded.

The obvious lower bound on $c(P_n)$ is implied by the maximum load and equals $\lceil n/2 \rceil \lfloor n/2 \rfloor$. In order to provide a close upper bound we propose a simple scheme that uses $\frac{5}{16}n^2 + n/2$ colors.

Let us consider first the case in which n is a multiple of 4. The scheme overlaps the receiving intervals of nodes $n/2 + i$ and $n - 1 - i$ for i between 0 and $n/4 - 1$, each such pair corresponding to nodes at distance $n/2 - 2i - 1$ and thus realizing the saving of $n/2 - 2i - 1$ colors, for a total of $\sum_{i=0}^{n/4-1}(n/2 - 2i - 1) = n^2/16$ reused colors. It is easy to verify that for each i such a pair allows a complete overlapping of the receiving interval of node $n/4 + i$. Furthermore, if $n/4 + i = 2j$ for a given integer j, that is $n/4 + i$ is an even number, then the receiving interval of j can be completely overlapped with the interval of $2j$; proceeding in this way, if j is even, the interval of $j/2$ can completely overlapped with the one of j and so forth, until an odd number is obtained. Therefore, after all the possible pairs $n/2 + i$ and $n - 1 - i$ have been considered, the receiving interval of each node j smaller than $n/4$ is completely overlapped with the interval of node $2j$. An example of overlapping for a given pair is given in Fig. 5, while Fig. 6 gives a complete construction for a path of 15 nodes.

The number of colors used in the above scheme is thus equal to the cardinality of the set given by the union of the receiving intervals of the nodes from $n/2$ to $n - 1$. Subtracting

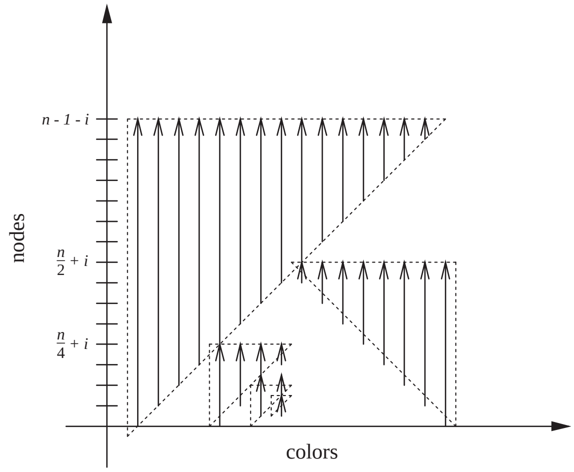

Fig. 5. The circular spectrum construction for a given pair.

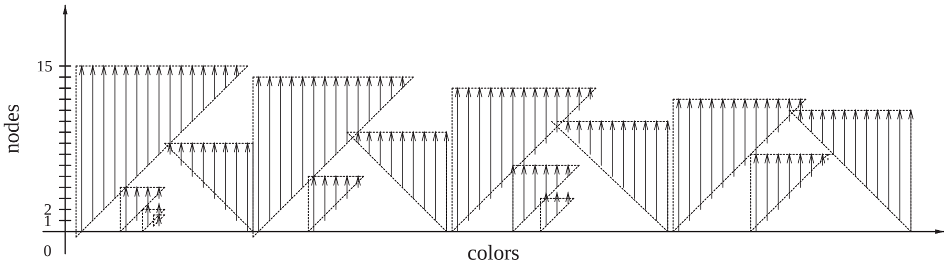

Fig. 6. The circular spectrum construction for P_{16}.

the saving of $n^2/16$ colors from the quantity $\sum_{i=n/2}^{n-1} i = 3n^2/8 - n/4$, that is, the sum of the sizes of all the intervals from $n/2$ to $n-1$, we finally get a total number of colors equal to $(5n^2/16) - n/4$.

If n is not a multiple of 4, an analogous construction can be obtained by overlapping the receiving intervals of all the possible pairs of the nodes $\lfloor n/2 \rfloor + i$ and $n-1-i$ and then completely overlapping the intervals of the nodes before $\lfloor n/2 \rfloor$ similarly as above. Notice that there are $\lfloor (n - \lfloor n/2 \rfloor)/2 \rfloor = \lfloor \lceil n/2 \rceil /2 \rfloor$ such pairs, and if there is an odd number of nodes between $\lfloor n/2 \rfloor$ and $n-1$, the intermediate node $\lfloor 3n/4 \rfloor$ does not belong to any pair and its receiving interval is left completely disjoint from all the others. Since each pair $\lfloor n/2 \rfloor + i$ and $n-1-i$ corresponds to nodes at distance $\lceil n/2 \rceil - 2i - 1$ and thus realizes the saving of $\lceil n/2 \rceil - 2i - 1$ colors, by similar arguments it follows that the total number of used colors is $\sum_{i=\lfloor n/2 \rfloor}^{n-1} i - \sum_{i=0}^{\lfloor \lceil n/2 \rceil /2 \rfloor -1} (\lceil n/2 \rceil - 2i - 1) \leqslant (5n^2/16) + n/2$.

Therefore, the following theorem has been proved.

Theorem 3.3. $c(P_n) \leqslant 5n^2/16 + n/2$.

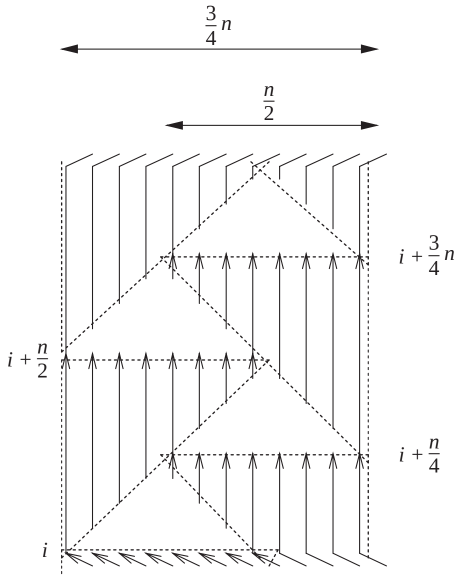

Fig. 7. An example of overlapping in C_{16} in the circular spectrum construction.

4. Gossiping in cycles

In a cycle network C_n of order n, nodes are numbered from 0 to $n-1$ and are arranged on a circle so that each i, $0 \leqslant i \leqslant n-1$, is adjacent to $(i+1) mod\, n$.

The lower bound on $l(C_n)$ and $c(C_n)$ coming from the maximum load is $\lceil (\lfloor n/2 \rfloor \lceil n/2 \rceil)/2 \rceil$, as the cycle can be divided in two parts (of $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ nodes, respectively) that can communicate only through the two separating edges.

In the following, we will focus only on the circular spectrum model, as for the linear case our constructions could not improve on the one that can be directly inferred from paths not exploiting the adjacency of nodes 0 and $n-1$.

Let us first consider n a multiple of 4.

To minimize the overall load, our routing follows shortest paths. Thus, each node receives $n/2-1$ colors anti-clockwise and the remaining $n/2$ colors clockwise. We just show the construction for the clockwise direction, since the other one is analogous and uses fewer colors as the node $i+n/2$ (diametrically opposite to i) is always reached clockwise.

The nodes of C_n can be partitioned into quadruples $(i, i+n/4, i+n/2, i+3n/4)$ for $i=0, \ldots n/4-1$. For a given value of i, the receiving intervals of the nodes in the quadruple are pairwise identical $I_i = I_{i+n/2}$ and $I_{i+n/4} = I_{i+3n/4}$. Moreover, the two pairs share $n/4$ colors (see Fig. 7).

Thus, the total number of colors used by the quadruple is $3n/4$, and as there are $n/4$ quadruples, the total number of colors used by the construction is $3n^2/16$.

If n is not a multiple of 4, we form the quadruples $(i, i+\lfloor n/4 \rfloor, i+\lfloor n/2 \rfloor, i+\lfloor n/2 \rfloor + \lfloor n/4 \rfloor)$ for $i=0, \ldots, \lfloor n/4 \rfloor - 1$, leaving at most 3 nodes not belonging to any quadruple and whose receiving intervals, each of size $\lfloor n/2 \rfloor$, are completely disjoint from all the other ones. By a similar argument, each quadruple uses $2\lfloor n/2 \rfloor - \lfloor n/4 \rfloor$ colors, so that the total number of colors achieved by the construction is $\lfloor n/4 \rfloor\,(2\lfloor n/2 \rfloor - \lfloor n/4 \rfloor) + 3\lfloor n/2 \rfloor \leqslant (3n^2/16) + 3n/2$.

Therefore, the following theorem holds.

Theorem 4.1. $c(C_n) \leqslant (3n^2/16) + 3n/2$.

5. Conclusions

We have given upper and lower bounds on the number of colors needed to perform all-to-all communications in all-optical networks using switching elements able to route single wavebands (intervals of colors).

Linear spectrum routers may not be applicable in all cases, as there are topologies in which they are not able to realize given communication tasks. The increase in the number of colors used for all-to-all communications in paths and cycles is, respectively, at most twice and four times those achieved with standard all-optical devices [3,17,8].

The circular spectrum elements are universal in the sense that they can be used to realize any communication task on any topology. Moreover, in paths and cycles the numbers of used colors for all-to-all communications are, respectively, at most $\frac{5}{4}$ and $\frac{3}{2}$ times those reachable by standard all-optical devices, as it can be checked by a direct comparison with the maximum load. Therefore, only a slight increase in the number of colors is needed, with the benefit of a substantial reduction of the hardware complexity and costs.

A natural open problem is that of reducing the gap between the lower and upper bounds on the number of colors for single-band routers in the circular case.

Another interesting research direction for both types of spectrum models is extending our results to routers that are able to route up to k bands instead of one. In such a setting, it would be nice to find close upper and lower bounds on the number of colors needed to perform routing on trees, as compared to the maximum load. We note that (for $k = 1$) while under the assumption of linear spectrum, routing on trees is not always possible, in the case of the circular spectrum model there are asymptotically matching lower and upper bounds on the ratio between number of colors $c(G)$ and maximum load $load(G)$. Is it possible to reduce the (constant) multiplicative gap? What about communication patterns different from the all-to-all?

As already observed, significantly different results hold for standard unrestricted routers, where the all-to-all communications on trees can be performed using a number of colors equal to the maximum load $load(G)$ [8], while for any communication pattern the number of needed colors is always at most $\frac{5}{3}$ times $load(G)$ [12].

It also may be interesting to explore these routing constraints for multi-hop connections.

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References

[1] B. Beauquier, J.C. Bermond, L. Gargano, P. Hell, S. Perennes, U. Vaccaro, Graph problems arising from wavelength-routing in all-optical networks, in: Proc. 2nd Workshop on Optics and Computer Science, Part of IPPS, 1997.

[2] C. Berge, Graphs, Vol. 6, North-Holland Mathematical Library, 1985.

[3] J.C. Bermond, L. Gargano, S. Perennes, A.A. Rescigno, U. Vaccaro, Efficient collective communication in optical networks, in: Proc. 23rd Internat. Coll. on Automata, Langauges and Programming, ICALP, 1996, pp. 574–585.

[4] C.A. Brackett, Dense wavelength division multiplexing networks: principles and applications, IEEE J. Select. Areas Commun. 8 (6) (1990) 948–964.

[5] N.K. Chung, K. Nosu, G. Winzer, Special issue on dense WDM networks, IEEE J. Select. Areas Commun. 8 (6) (1990) 1214–1215.

[6] D.H.C. Du, R.J. Vetter, Distributed computing with high-speed optical networks, Computer 26 (2) (1993) 8 –18.

[7] T. Erlebach, K. Jansen, Scheduling of virtual connections in fast networks, in: Proc. 4th Workshop on Parallel Systems and Algorithms (PASA), 1996, pp. 13–32.

[8] L. Gargano, P. Hell, S. Perennes, Coloring all directed paths in a symmetric tree with an application to optical networks, J. Graph Theory 38 (4) (2001) 183–196.

[9] L. Gargano, U. Vaccaro, Routing in all-optical networks: algorithmic and graph—theoretic problems, in: I. Althofer et al. (Eds.), Numbers, Information and Complexity, Kluwer Academic, Dordrecht, 2000, 555–578.

[10] P.E. Green, Fiber-Optic Communication Networks, Prentice-Hall, Englewood Cliffs, NJ, 1992.

[11] H.S. Hinton, Architectural considerations for photonic switching networks, IEEE J. Select. Areas Commun. 6 (7) (1988) 1209–1226.

[12] C. Kaklamanis, P. Persiano, T. Erlebach, K. Jansen, Constrained bipartite edge coloring with applications to wavelength routing, in: Proc. 24th Internat. Coll. on Automata, Languages and Programming (ICALP), 1997, pp. 493–504.

[13] I.A. Karapetian, On coloring of arc graphs, in: Docladi (Reports) of the Academy of Science of the Armenian Soviet Socialist Republic, Vol. 70(5), 1980, pp. 306–311.

[14] R. Klasing, Methods and problems of wavelength-routing in all-optical networks, in: Proc. of the MFCS'98 Workshop on Communication, 1998, pp. 1–9.

[15] S. Personick, Review of fundamentals of optical fiber systems, IEEE J. Select. Areas Commun. 1 (3) (1983) 373–380.

[16] R. Ramaswami, Multi-wavelength lightwave networks for computer communication, IEEE Commun. Magazine 31 (2) (1993) 78–88.

[17] G. Wilfong, Minimizing wavelengths in an all-optical ring network, in: Proc. 7th Internat. Symp. on Algorithms and Computation, ISAAC, 1996, pp. 346–355.